$$2\left(T-\sum_{k=1}^{n} a_kT_k^*-\sum_{k=1}^{n} b_kT_k^\circ\right)\rho^{-1}\mathbf{n}=\mathbf{N}+\mathbf{\Phi}-\sum_{k=1}^{n} c_k\mathbf{\Phi}_k \qquad (c_k=a_k+b_k) \qquad (20)$$

Since $\mathbf{N}$ is chosen orthogonal to $\mathbf{\Phi}$ and $\mathbf{\Phi}_k$, from (11) and (13) we find that $\mathbf{N}=0$. Moreover, from Eq. (20) it then follows that

$$\mathbf{\Phi}=\sum_{k=1}^{n} c_k\mathbf{\Phi}_k$$

From the results obtained we have, as particular cases, the corresponding theorem for the points of constant mass [2] and the proper Bonnet theorem [1]. Just as it was done in [2], the results obtained can be applied to the study of motion of points of variable mass in a gravitational field of two fixed centers.

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ON THE MEMBRANE STATE OF MULTIPLY-CONNECTED CONVEX SHELLS

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P. I. KUDRIK
(Kiev)

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Conditions for the realization of the membrane state of equilibrium of multiply-connected shells of positive Gaussian curvature subjected to surface and edge forces, are investigated; the concepts of correctness and stability of the membrane states are introduced. The terminology and notation correspond to that used in [1, 2].

1. Let the middle surface S of a multiply-connected shell of positive Gaussian curvature be referred to an isometrically conjugate curvilinear coordinate system x^1, x^2 and let us write its equation in the vector form $\mathbf{r}=\mathbf{r}(x^1, x^2)$. Relative to the regularity of the shell we assume that $S\in D_{k+3,p}$, $p>2$, $k\geqslant 0$. The middle surface S and its outline $L=L_0+L_1+\ldots+L_m$ in the coordinate plane $\zeta=x^1+ix^2$ are a domain G with the boundaries $\Gamma=\Gamma_0+\Gamma_1+\ldots+\Gamma_m$ in a homeomorphic way. The lines of the holes in the shell $L_0, L_1, \ldots, L_m$ are closed, three-dimensional, nonreentrant curves of the Liapunov class. A x^1, x^2 coordinate system can always be found so that the point $\zeta=0$ would belong to the interior of the domain G and the contour Γ_0 would enclose all the other curves $\Gamma_1, \ldots, \Gamma_m$. Finally, the second quadratic form of the surface S in the isosymmetrically conjugate x^1, x^2 coordinate system has the canonic form: $\sqrt{gK}\,[(dx^1)^2 + (dx^2)^2]$, i.e. the coefficients of this form are connected by the relationships: $b_{11} = b_{22} = \sqrt{gK}$, $b_{12} = b_{21} = 0$. Here g and K are, respectively, the discriminant of the first quadratic form of the surface S and its Gaussian curvature.

In tensor notation the system of equations of membrane shell theory is the following [3]:

$$\sqrt{gK}\,(T^{11} + T^{22}) + Z = 0$$

$$\frac{\partial T^{\alpha\beta}}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\lambda} T^{\lambda\beta} + \Gamma^\beta_{\alpha\gamma} T^{\alpha\gamma} + X^\beta = 0 \qquad (\beta = 1, 2)$$

where $T^{\alpha\beta}$ is the contravariant stress resultant tensor, X^β are the contravariant components of the surface load X, Z is its normal component, $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol of the second kind for the surface S. If the stress resultant T^{22} is eliminated from this system and the complex stress function is introduced by means of $W(\zeta) = \sqrt{g}\,(T^{11} - iT^{12})$, then we obtain an equation of Carleman type for it

$$\frac{\partial}{\partial\bar\zeta} W - A(\zeta) W - \overline{B(\zeta)}\,\overline{W} = F(\zeta), \qquad \zeta \in G \tag{1.1}$$

Here

$$\frac{\partial}{\partial\bar\zeta} W = \frac{1}{2}\left(\frac{\partial W}{\partial x^1} + i\,\frac{\partial W}{\partial x^2}\right)$$

$$A = \frac{1}{4}(\Gamma_{22}^1 - \Gamma_{11}^1 - 2\Gamma_{12}^2) + \frac{i}{4}(\Gamma_{11}^2 - \Gamma_{22}^2 - 2\Gamma_{12}^1)$$

$$B = \frac{1}{4}(\Gamma_{22}^1 - \Gamma_{11}^1 + 2\Gamma_{12}^2) - \frac{i}{4}(\Gamma_{11}^2 - \Gamma_{22}^2 + 2\Gamma_{12}^1)$$

$$F = \frac{1}{2}\sqrt{K}\,\frac{\partial}{\partial\zeta}\left(\frac{Z}{K}\right) - \frac{\sqrt{g}}{2}(X^1 - iX^2)$$

Let us seek the solution of (1.1) in the class of generalized analytic functions continuous in $G + \Gamma$. It has been shown in [3] that if a stress resultant $\mathbf{T}_L$ is applied along the contour L of a membrane shell, then it is expressed in terms of the boundary values of the desired function W by means of the formula

$$\mathbf{T}_L = \mathrm{Re}\left[\frac{1}{i} W(\tau)\frac{d\tau}{ds}(\mathbf{r}_1 + i\mathbf{r}_2)\right] - \frac{Z}{2\sqrt{K}}\mathrm{Re}\left[\frac{1}{i}\,\frac{d\tau}{ds}\,(\mathbf{r}_1 + i\mathbf{r}_2)\right] \qquad \tau \in \Gamma \tag{1.2}$$

Here s is a natural parameter of the contour L and $\mathbf{r}_1$, $\mathbf{r}_2$ are vectors of the fundamental basis on the middle surface S.

Let $\mathbf{n}$ and $\mathbf{s}$, respectively, denote the directions of the normal at an arbitrary point of the surface S and of the tangent to the contour L. Then the vector $\mathbf{l} = [\mathbf{s}\times\mathbf{n}]$ becomes the direction of the tangential normal along the shell contour L.

2. We consider the following problem from shell theory. Let a specified surface load $\mathbf{X} = X^\alpha \mathbf{r}_\alpha + Z\mathbf{n}$ and some edge stress resultant $\mathbf{T}_L$ relative to which only the component in the direction λ, forming an angle $\varphi(s)$ with the dirction of the tangent s to the line L is known, act on the shell. Without determining the second component of the stress resultant $\mathbf{T}_L$, it must be established whether the shell is in the membrane state under the effect of external forces given in such a manner. These problems are often encountered in engineering. In practice, the fact that stress concentrations originate in a definite zone near the hole in a shell, for whose determination there are a number of effective methods [4], is certainly taken into account.

We shall show that the formulated problem reduces to the solution of a Riemann-Hilbert boundary value problem in the class of generalized analytic functions. Indeed, let us multiply both sides of (1.2) scalarly by λ and let $f = T_L\lambda$ denote a given component of the edge stress resultant T_L. Then (1.2) is converted into the form

$$\operatorname{Re}\left[W \frac{d\tau}{ds}\frac{d\tau}{d\lambda'}\right] = h, \qquad \tau \in \Gamma \tag{2.1}$$

$$h = -\frac{1}{\sqrt{g}} f - \frac{Z}{2\sqrt{K}} \operatorname{Re}\left[\frac{d\tau}{ds}\frac{d\tau}{d\lambda'}\right]$$

Here λ' is the unit vector conjugate to λ which is tangent to the middle surface S, where $[\lambda\lambda'] = \mathbf{n}$. The formula

$$\frac{\sqrt{g}}{i}\frac{d\tau}{d\lambda'} = \lambda_1 + i\lambda_2$$

was used to derive (2.1), where λ_1, λ_2 are covariant components of λ.

The following Riemann-Hilbert boundary value problem has therefore been obtained: find the generalized analytic function W from (1.1) which is continuous in $G + \Gamma$ and satisfies the boundary condition (2.1) on the boundary Γ of the domain G.

Let us draw upon the auxiliary, conjugate Riemann-Hilbert boundary value problem to investigate this boundary value problem: find the generalized analytic function U satisfying the equation in the domain G

$$\frac{\partial}{\partial \bar{\zeta}} U + A(\zeta) U + B(\zeta)\bar{U} = 0, \qquad \zeta \in G \qquad (U = u_1 + iu_2)$$

which is continuously extendable on the boundary Γ and satisfies the condition on the boundary Γ itself

$$\operatorname{Re}\left[\frac{d\tau}{d\lambda'} U(\tau)\right] = 0, \qquad \tau \in \Gamma \tag{2.3}$$

The following geometric interpretation can be given to the boundary value problem (2.2), (2.3): this is the problem of finding infinitesimal first order bending for the surface S when covariant components of the displacement field u_1, u_2 along the contour L are connected by the relationship (2.3).

3. Let n, χ and n', χ', respectively, denote the number of linearly independent solutions and the indices of the boundary value problems (1.1), (2.1) − (2.3). We use the Vekua theorems (see [3], pp. 252-257) in application to the boundary value problems formulated above. The equality $n - n' = \chi - \chi' = 2\chi + 1 - m$ thus holds, where the integer m indicates the order of connectedness of the shell. If the index of the problem (1.1), (2.1) is negative $\chi < 0$, it then admits of solution only when the so-called solvability conditions are satisfied

$$\int_L f U_\lambda^{(j)}\, ds + \iint_S \mathbf{U}^{(j)}\mathbf{X}\, dS = 0 \qquad (j = 1, \ldots, n') \tag{3.1}$$

where $\mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(n')}$ is a complete system of linearly independent displacement fields of the middle surface for its infinitesimal bending.

Now, let us compute the indices of the boundary value problems under investigation. We have

$$\chi = m - 1 - \chi_{\lambda'}, \quad \chi' = \chi_{\lambda'}, \quad \chi_{\lambda'} = \operatorname{Ind}\frac{d\tau}{d\lambda'}$$

Therefore, when $\chi_{\lambda'} < 0$, then $n' = 0$, $n = m - 1 - 2\chi_{\lambda'}$. Hence it follows that the problem (1.1), (2.1) is always solvable, and its solution is given by

$$W(\zeta) = W_0(\zeta) + \sum_{k=1}^{m-1-2\chi_{\lambda'}} C_k W_k(\zeta)$$

Here C_k are arbitrary real constants, $W_1, \ldots W_{m-1-2\chi_{\lambda'}}$ are linearly independent solutions of the homogeneous problem corresponding to (2.1), (1.1), and W_0 is a particular solution of the inhomogeneous problem.

The form of the solution of the boundary value problem shows that $m - 1 - 2\chi_{\lambda'}$ membrane states of equilibrium can be realized for a given system of external forces in the shell. We shall say about such a situation that the physical problem is posed quasi-correctly. However, by imposing additional conditions of point character it can always be achieved that just one definite membrane state would originate in the shell. For this it is sufficient to give the value of two components of the stress resultant vector $\mathbf{T}_L$ at k internal points $m_1, m_2, \ldots, m_k$ of the shell and the value of one component of the stress resultant $\mathbf{T}_L$ at k' points $m_1', m_2', \ldots, m_{k'}'$ on the contour L. The following three conditions should hence be conserved: (1) An odd number of points must be taken on each contour L_j $(j = 1, \ldots, m)$; (2) $2k + k' = 2n + 1 - m$; (3) $k' \geqslant m$. Then, as the research of Vekua in [3] shows, the Riemann-Hilbert boundary value problem (1.1), (2.1) has a unique solution and the problem itself is correct. Correctness of the problem means that the shell will work in a definite membrane state for insignificant changes in the shell geometry (its shape and size, Gaussian curvature, hole locations, etc.) and variations in the external forces. This circumstance implies a unique stability of the shell membrane properties.

Now, let $\chi_{\lambda'} > m - 1$, then $n = 0$, $n' = 2\chi_{\lambda'} + 1 - m$, and the boundary value problem (1.1), (2.1) has a negative index. Therefore, $2\chi_{\lambda'} + 1 - m$ conditions of the form (3.1) should be satisfied for its solvability. It can happen that only trivial bending fields enter into (3.1); then we obtain the ordinary shell equilibrium conditions which hold by the assumption of the problem. If the shell is weakened by more than five holes, then there will certainly enter at least one nontrivial bending field into (3.1), and therefore, the realization of the membrane state in the shell will not always be possible.

Finally, let us examine the case when $0 \leqslant \chi_{\lambda'} \leqslant m - 1$. This is possible only when the shell is weakened by two or more holes $(m > 0)$. For $\chi_{\lambda'} = m - 1$ there are two possibilities: (1) $n = 0$ and (2) $n = 1$. Then correspondingly: (1) $n' = m - 1$ and (2) $n' = m$. This means that if the shell is weakened by more than two holes $(m > 1)$, then the boundary value problem (1.1), (2.1) is not correct in both cases. But a shell with two holes $(m = 1)$ always realizes a membrane state (and moreover unique) in the first case $(n = 0)$ and the corresponding problem is not correct in the second case $(n = 1)$.

4. Let us call the two directions λ and λ^* tangent to the surface S directions of the same class if $\chi_\lambda = \chi_{\lambda^*}$. If the angle φ^* between the two vectors λ and λ^* at each point of the contour L is such that $|\varphi^*| < \pi$, then these vectors evidently belong to one class. For example, if the vector λ belongs to the class of tangents s to the contour L, then $\chi_\lambda = \chi_{\lambda^*} = \chi_s = 1 - m$. Moreover, if the angle φ^* is a Hölder-continuous function: $\varphi^* \in C_\alpha(L)$ and its norm in the metric $C_\alpha(L)$ satisfies the condition

$\| \varphi - \varphi^* \|_{C_\alpha} < \varepsilon$, where ε is a sufficiently small positive number, then it is said that the direction λ^* is a normal perturbation of the direction λ.

In addition to the problem (2.1), (1.1), let us consider the normally perturbed problem of the form

$$\frac{\partial}{\partial \bar{\zeta}} W - A^*(\zeta) W - \overline{B^*(\zeta)} \overline{W} = F^*(\zeta), \qquad \zeta \in G$$

$$\operatorname{Re}\left[W \frac{d\tau}{ds} \frac{d\tau}{d\lambda^*}\right] = h^*$$

$$\| A - A^* \|_{L_p} < \varepsilon, \quad \| B - B^* \|_{L_p} < \varepsilon, \quad \| F - F^* \|_{L_p} < \varepsilon$$

$$\| \varphi - \varphi^* \|_{C_\alpha} < \varepsilon, \quad \| h - h^* \|_{C_\alpha} < \varepsilon$$

It is clear that if the boundary problem (1.1), (2.1) is quasi-correct, then the normally perturbed problem is also quasi-correct for sufficiently small ε.

In conclusion, let us examine the case when the vector λ belongs to the class s. Then $\varkappa = 2(m-1)$. Therefore, $\chi = -2$, $n = 0$, $n' = 3$ for simply connected shells. This means that three conditions of the form (3.1) should be satisfied in order to realize a membrane state in a shell with one hole. If the contour L of the middle surface passes along an isometrically conjugate line, then the corresponding surface bendings will be trivial [2] and the formulated problem has a unique solution. For shells with three or more holes the membrane state will be quasi-correct since $n = 0$ and $n' = 3m - 3$. For doubly connected shells of positive curvature the membrane state cannot always be realized; however, if the hole contours of the shell coincide with isometrically conjugate lines on the middle surface, then such a state is realized unconditionally.

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ADDENDUM TO THE PAPER "ON CERTAIN EXACT SOLUTIONS OF THE FOURIER EQUATION FOR REGIONS VARYING WITH TIME"

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G. A. GRINBERG and V. A. KOSS
(Leningrad)
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The exact solutions given in [1] are generalized to the case of cylindrical and spherical sectors rotating about the azimuth relative to the coordinate origin either at a uniform rate or with uniform acceleration (or deceleration). The